

QUANTUM COSMOLOGY WITH $R + R^2$ GRAVITY

A.K.Sanyal *

Department of Physics, Jangipur College, Murshidabad, 742213
and

Relativity and Cosmology Research Centre, Department of Physics, Jadavpur University, Kolkata 700032
B.Modak †

Department of Physics, Kalyani University, 741235, India
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Canonical quantization of an action containing curvature squared term requires introduction of an auxiliary variable. Boulware et al [1] prescribed a technique to choose such a variable, by taking derivative of the action with respect to the highest derivative of the field variable, present in the action. It has been shown that [2] this technique can even be applied in situations where introduction of auxiliary variables are not at all required, leading to wrong Wheeler-deWitt equation. It has also been pointed out that [2] Boulware et al's [1] prescription should be taken up only after removing all possible total derivative terms from the action. Once this is done only a unique description of quantum dynamics would emerge. For curvature squared term this technique yields, for the first time, a quantum mechanical probability interpretation of quantum cosmology, and an effective potential whose extremization leads to Einstein's equation. We conclude that Einstein-Hilbert action should essentially be modified by atleast a curvature squared term to get a quantum mechanical formulation of quantum cosmology and hence extend our previous work [2] for such an action along with a scalar field.

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I. INTRODUCTION

Quantum cosmology is most elusive as the role of time [3] is not unique and one is being unable to define the Hilbert space. This is due to the fact that 'time' is not an external parameter in general theory of relativity, rather it is intrinsically contained in the theory, unlike its role in quantum mechanics or quantum field theory in flat space time. In curved space time, different slices correspond to different choices of time leading to inequivalent quantum theories. Likewise, the canonical quantization of gravity is devoid of an unique time variable and hence the definition of probability of emergence of a particular Universe out of an ensemble is ambiguous. The canonical quantization of Einstein-Hilbert action together with some matter fields yields the Wheeler-deWitt equation which does not contain time a priori, although it emerges intrinsically through the scale factor of the Universe. However, if the canonical variable is so chosen that one of the true degrees of freedom is disentangled from the kinetic part of the canonical variables, then this kinetic part in the corresponding quantum theory yields a quantum mechanical flavour of time. This is possible only if the Einstein-Hilbert action is replaced by curvature squared action or modified by the introduction of curvature squared term, in the Robertson-Walker minisuperspace model. In a recent publication [2] we have presented such a choice of the canonical variable in a curvature squared action whose quantization yields a Schrödinger like equation. Further, as a consequence, we could identify the nature of the space and time like variables in the Robertson-Walker minisuperspace model from the equation of continuity and hence could establish the idea of the probability and the current densities in the context of quantum cosmology.

The relevance of the fourth order gravity theory in cosmology was first revealed by Starobinsky [4], though it appeared earlier in the context of quantum field theory in curved space time. Starobinsky [4] presented a solution of the inflationary scenerio without invoking phase transition in the very early Universe, from a field equation containing only the geometric terms. However, the field equations could not be obtained from the action principle, as the terms in the field equations are generated from the perturbative quantum field theory. Later, Starobinsky and Schmidt [5] have shown that the inflationary phase is an attractor in the neighbourhood of the fourth order gravity theory. Such wonderful relevance of higher order gravity theory in the context of cosmology inspired some authors to give

*e-mail:aks@juphys.ernet.in

†e-mail:bmodak@klyuniv.ernet.in

interpretation of the quantum cosmological wavefunction with R^2 [6] and even R^3 [7] terms in the Einstein-Hilbert action.

Further the functional integral for the wavefunction of the Universe proposed by Hartle-Hawking [8] runs into serious problem, since the wavefunction diverges badly. There are some prescriptions [9] to avoid such divergences, though a completely satisfactory result has not yet been obtained. However, to get a convergent functional integral, Horowitz [10] proposed an action in the form

$$S = \int d^4X \sqrt{g} [AC_{ijkl}^2 + B(R - 4\lambda)^2] \quad (1)$$

where C_{ijkl} is the Weyl tensor, R is the Ricci scalar, λ is the cosmological constant and A, B are the coupling constants. The action (1) reduces to the Einstein-Hilbert action at the weak energy limit [10]. To obtain a workable and simplified form of the field equations, one may consider a spatially homogeneous and isotropic minisuperspace background, for which the Weyl tensor trivially vanishes. In a previous paper [2], we considered the action (1) retaining only the curvature squared term. The field equations for such an action can be obtained by the standard variational principle. In the variational principle, the total derivative terms in the action are extracted and one gets a surface integral which is assumed to vanish at the boundary or the action is chosen in such a way that those surface integral terms have no contribution. However, for canonical quantization this principle is not of much help and one has to express the action in the canonical form, which is achieved only through the introduction of the auxiliary variable. Auxiliary variable can be chosen in an adhoc manner, and different choice of such variable would lead to different description of quantum dynamics, keeping the classical field equations unchanged. In view of this Ostrogradski [11] in one hand and Boulware et al [1] on the other, have made definite prescriptions to choose such variables. Ostrogradski's prescription [11] was followed by Schmidt [12]. We shall consider Boulware et al's [1] prescription in which it was proposed that the auxiliary variable should be chosen by taking the first derivative of the action with respect to the highest derivative of the field variable present in the action.

Hawking and Luttrell [13] utilized Boulware et al's [1] technique to identify the new variable and showed that the Einstein-Hilbert action along with a curvature squared term reduces to the Einstein-Hilbert action coupled to a massive scalar field, assuming the conformal factor. Horowitz [10] on the other hand, showed that the canonical quantization of the curvature squared action yields an equation which is similar to the Schrödinger equation. Pollock [14] also used the same technique to the induced theory of gravity and obtained the same type of result as that obtained by Horowitz [10], in the sense that the corresponding Wheeler-deWitt equation looks similar to the Schrödinger equation.

The striking feature of Boulware et al's [1] prescription is that it can even be applied in situations where introduction of the auxiliary variable is not at all required, eg. in the induced theory of gravity, vacuum Einstein-Hilbert action etc. It is not difficult to see that the classical field equations remain unchanged with or without the introduction of the auxiliary variables. Now the question is whether quantum dynamics remain unaffected? Putting other way round, the question turns out to be whether the solution of the Schrödinger like equation obtained by Pollock [14] in the induced theory of gravity by introducing auxiliary variable will satisfy the Wheeler-deWitt equation corresponding to the same action that can be obtained without introducing such variable? The answer is simply no. We have shown [2] that the introduction of an auxiliary variable does not change the classical field equations in situation mentioned above, however, the quantum dynamics are different. The vacuum Einstein-Hilbert action in the background of Robertson-Walker metric has been chosen for this purpose. We have shown in such a toy model [2], 'toy', since it is not required to follow Boulware et al's [1] prescription in the vacuum Einstein-Hilbert action, that the introduction of an auxiliary variable leads to wrong Wheeler-deWitt equation. So what went wrong with Boulware et al's [1] prescription? This is what we have concluded in [2], that before taking up the said prescription, one has to eliminate all possible total derivative terms from the action. Once this is done, it would not at all be possible to introduce auxiliary variables in situations mentioned above, where such introduction is unnecessary, and hence only a unique and correct quantum dynamics would emerge. In this sense, the auxiliary variable identified by Hawking and Luttrell [13] and Schrödinger like equation obtained by Horowitz [10] and Pollock [14] are wrong. We [2] have obtained the correct quantum dynamics for curvature squared action and showed that the continuity equation identifies the nature of space and time like variables in the minisuperspace, establishing, perhaps for the first time the quantum mechanical idea of the probability and the current densities in quantum cosmology. It has also been shown that the correct description of the curvature squared action leads to an effective potential, whose extremum yields vacuum Einstein's equation, which is a desirable feature of higher derivative gravity theory, in the weak energy limit. The emergence of the Schrödinger like equation, the quantum mechanical concept of the probability density and the effective potential lead us to conclude that the quantum description of cosmology is incomplete without atleast a curvature squared term in the action. This is the reason why in this paper we have taken up the task of generalizing our previous work [2] by modifying the Einstein-Hilbert action with a curvature squared term along with a matter field.

As already mentioned, in order to obtain the correct and unique quantum cosmological description of a system whose action contains curvature squared term, one has to first remove all possible total derivative terms from the action and

then follow Boulware et al's [1] prescription. The whole formulation requires four steps, viz. 1) remove all possible total derivative terms from the action, 2) introduce auxiliary variable, which is the first derivative of the action with respect to the highest derivative of the field variable present in the action, 3) cast the action in the canonical form and obtain classical field equations, 4) introduce basic variables to canonically quantize the Hamiltonian constraint equation. At this stage we would like to mention that the Hamiltonian in our formalism before introducing basic variables [2] turned out to be the same as that obtained by Schmidt [12] following Ostrogradski's [11] prescription. This is due to the fact that, in that prescription [11] total derivative terms do not appear. However, the importance of removal of the total derivative terms has not been stressed, the method of introducing auxiliary variables and process of casting the action in the canonical form is not so vivid and elegant as in Boulware et al's [1] formalism and quantization procedure by introducing basic variables has not been mentioned in that approach [11]. Therefore we have selected the other method.

This paper has been organized in the following manner. In section 2. we present the field equation for an action containing curvature and curvature squared term along with a minimally coupled matter field, following the proposal mentioned above. Further we cast the action in canonical form with respect to the basic variables, which shows the appearance of a Hamiltonian $-P_\alpha$ of a system described by the basic variables and their conjugate momenta, though the actual Hamiltonian H ie. the super Hamiltonian constraint to vanish. The form of the above action also identifies the time variable, which is the intrinsic time, whose canonical momentum is P_α , or the Hamiltonian $-P_\alpha$ can be thought of as the generator of the translation along the above time variable. The action written in this form indicates the true dynamical degree of freedom, disentangled from the kinetic variable ie. the time variable. We claim that the Hamiltonian $(-P_\alpha)$ is the correct Hamiltonian in the sense that it contains the basic variables and their conjugate momenta and its canonical quantization yields the Wheeler-deWitt equation which mimics the Schrödinger equation, along with an effective potential.

In section 3. we present the conventional continuity equation of quantum mechanics from the Schrödinger like Wheeler-deWitt equation. The continuity equation in quantum cosmology was first presented by Sanyal and Modak [2] only in a recent publication and it is interesting to note that, as the probability and current densities are finite, so also the wavefunction and its derivatives are finite throughout the evolution of the Universe if and only if there are no singularities in the domain of quantum cosmology.

Section 4. has been devoted to extract physical implication of the effective potential V_e . There are singularities in the effective potential. It is the function of the scalar field and the expansion parameter, which is the Hubble parameter of the classical field equations and it is diverging both at the zero's and at the very large values of the expansion parameter. The effective potential is asymmetric with respect to the expansion and collapse of the Universe and its extremum yields classical equations.

Section 5. is devoted in making some concluding remarks.

II. CLASSICAL FIELD EQUATIONS AND QUANTIZATION

We consider the following action

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{16\pi G} (R - \frac{\beta}{6} R^2) + \frac{1}{2\pi^2} (\frac{1}{2} \phi_{,\mu} \phi^{,\mu} - V(\phi)) \right]. \quad (2)$$

Under the following form of the closed Robertson-Walker metric

$$ds^2 = e^{2\alpha(\eta)} [d\eta^2 - d\chi^2 - \sin^2\chi (d\theta^2 + \sin^2\theta d\phi^2)] \quad (3)$$

the action (2) can be expressed as

$$S = \int d\eta \left[\frac{3\pi}{4G} \{ (1 + \dot{\alpha}^2 + \ddot{\alpha}) e^{2\alpha} - \beta (1 + \dot{\alpha}^2 + \ddot{\alpha})^2 \} + \frac{1}{2} \dot{\phi}^2 e^{2\alpha} - V(\phi) e^{4\alpha} \right], \quad (4)$$

where, $\dot{\alpha} = \frac{d\alpha}{d\eta}$. According to our proposal, we remove all removable total derivative terms from the action, so that only lowest order terms appear in the action (4). Thus,

$$S = \int d\eta [M \{ (1 - \dot{\alpha}^2) e^{2\alpha} - \beta (1 + \dot{\alpha}^2)^2 - \beta \ddot{\alpha}^2 \} + \frac{1}{2} \dot{\phi}^2 e^{2\alpha} - V(\phi) e^{4\alpha}] + S_m, \quad (5)$$

where, $M = \frac{3\pi}{4G}$ and $s_m = M[\dot{\alpha} e^{2\alpha} - 2\beta(\dot{\alpha} + \frac{\dot{\alpha}^3}{3})]$.

At this stage, we define the auxiliary variable, which is

$$M\beta Q = -\frac{\partial S}{\partial \ddot{\alpha}} = 2M\beta\ddot{\alpha}, \text{ i.e. } Q = 2\ddot{\alpha}. \quad (6)$$

Introducing the auxiliary variable Q in action (5) and expressing the action in the canonical form, after removing the remaining total derivative terms, we obtain,

$$S = \int d\eta [M\{\beta\dot{Q}\dot{\alpha} + (1 - \dot{\alpha}^2)e^{2\alpha} - \beta(1 + \dot{\alpha}^2)^2 + \frac{\beta}{4}Q^2\} + \frac{1}{2}\dot{\phi}^2e^{2\alpha} - V(\phi)e^{4\alpha}] + S_{m_1}, \quad (7)$$

where, S_{m_1} is the surface integral, given by $S_{m_1} = S_m - \beta Q\dot{\alpha}$. Assuming S_{m_1} vanishes at the boundary, the classical field equations are found as,

$$\beta(\ddot{Q} - 12\ddot{\alpha}\dot{\alpha}^2 - 4\ddot{\alpha}) - 2e^{2\alpha}(1 + \dot{\alpha}^2 + \ddot{\alpha}) - \frac{e^{2\alpha}}{M}(\dot{\phi}^2 - 4V(\phi)e^{2\alpha}) = 0, \quad (8)$$

$$Q = 2\ddot{\alpha}, \quad (9)$$

$$\ddot{\phi} + 2\dot{\alpha}\dot{\phi} + e^{2\alpha}\frac{dV}{d\phi} = 0, \quad (10)$$

$$\beta(3\dot{\alpha}^4 + 2\dot{\alpha}^2 - \dot{\alpha}\dot{Q} + \frac{Q^2}{4} - 1) + (\dot{\alpha}^2 + 1)e^{2\alpha} = \frac{e^{2\alpha}}{M}(\frac{1}{2}\dot{\phi}^2 + V(\phi)e^{2\alpha}), \quad (11)$$

the last one being the Hamiltonian constraint equation. The definition of Q given in the equation (6) is thus recovered in equation (9). While equation (10) is the well known continuity equation of the scalar field ϕ , equations (8) and (11) are the correct classical field equations as it is evident by replacing Q by $2\ddot{\alpha}$. Now the Hamiltonian constraint equation in the phase space variables can be expressed as,

$$H = \frac{P_\alpha P_Q}{M\beta} + \frac{P_Q^4}{M^3\beta^3} + \frac{P_Q^2}{M\beta^2}(2\beta + e^{2\alpha}) + \frac{P_\phi^2 e^{-2\alpha}}{2} + M\beta - Me^{2\alpha} - \frac{M}{4}\beta Q^2 + V(\phi)e^{4\alpha} = 0, \quad (12)$$

where, P_α, P_Q and P_ϕ are the canonical momenta corresponding to α, Q and ϕ respectively, and $\dot{\alpha}, \dot{Q}$ and $\dot{\phi}$ are given by

$$\dot{\alpha} = \frac{P_Q}{M\beta}, \dot{\phi} = e^{-2\alpha}P_\phi, \dot{Q} = \frac{P_\alpha}{M\beta} + \frac{2(2\beta + e^{2\alpha})}{M\beta^2}P_Q + \frac{4P_Q^3}{M^3\beta^3}. \quad (13)$$

The Wheeler-deWitt equation is obtained through the quantization of the Hamiltonian constraint equation (12). Now as already mentioned, canonical quantization should be performed with the basic variables, viz. α and $\dot{\alpha}$. Let us choose $\dot{\alpha} = x$, and hence replace P_Q by $M\beta x$, in view of the first equation given in (13). Further, from equation (6), we get

$$M\beta Q = -\frac{\partial S}{\partial \ddot{\alpha}} = -\frac{\partial L}{\partial \ddot{\alpha}} = -\frac{\partial L}{\partial \dot{x}} = -P_x, \quad (14)$$

which means that Q should be replaced by $-\frac{P_x}{M\beta}$. From equation (14) it is evident that P_x is the canonical momentum corresponding to x . Hence equation (12) turns out to be

$$H = xP_\alpha - \frac{P_x^2}{4M\beta} + \frac{1}{2}P_\phi^2 e^{-2\alpha} + M\beta(1 + x^2)^2 + M(x^2 - 1)e^{2\alpha} + V(\phi)e^{4\alpha}. \quad (15)$$

As the Hamiltonian H is constraint to vanish, we get

$$-P_\alpha = -\frac{P_x^2}{4M\beta x} + \frac{1}{2x}e^{-2\alpha}P_\phi^2 + \frac{M\beta}{x}(1 + x^2)^2 + \frac{M}{x}(x^2 - 1)e^{2\alpha} + \frac{V(\phi)e^{4\alpha}}{x}. \quad (16)$$

We have already mentioned that P_α, P_ϕ and P_x are the canonical momenta corresponding to α, ϕ and x , respectively, however to understand the physical meaning of P_α , we write the action (7) in terms of the canonical coordinates, apart from a surface term, remembering $H = 0$, as

$$S = \int L d\eta = \int [P_\alpha \dot{\alpha} + P_Q \dot{Q} + P_\phi \dot{\phi}] d\eta \quad (17)$$

Further, from equation (14) we find

$$\dot{Q} = -\frac{1}{M\beta} \frac{dP_x}{d\eta} = -\frac{\dot{\alpha}}{M\beta} \frac{dP_x}{d\alpha}. \quad (18)$$

Now using the expressions for P_Q and \dot{Q} given by equations (13) and (18) respectively, the relations $d\alpha = \dot{\alpha} d\eta$, $\dot{\phi} = \frac{d\phi}{d\alpha} \dot{\alpha}$ in action (17) and finally rearranging terms, we obtain

$$S = \int [P_x \frac{dx}{d\alpha} + P_\phi \frac{d\phi}{d\alpha} + P_\alpha] d\alpha - \int \frac{d}{d\alpha} (x P_\alpha) d\alpha. \quad (19)$$

Apart from the surface term the action (19) looks similar to the conventional form of the action of a mechanical system described by the configuration space variables x, ϕ and their canonical conjugate momenta P_x, P_ϕ , with a Hamiltonian $(-P_\alpha)$, which is also the canonical momentum of α parametrized by the variable α , though the true Hamiltonian H of the gravitational system constraint to vanish. We know that the Hamiltonian of a system is the generator of the time translation. Since, $-P_\alpha$ in our system plays the role of the Hamiltonian, hence it acts as the generator of translation along α , and the variable α acts as the time variable. This feature is also consistent with the intrinsic concept of general theory of relativity, as time has no independent existence from geometry in describing gravitation, rather it is inbuilt in the theory, unlike situations encountered in the conventional classical and quantum mechanics, where time is an external parameter. The functional form of the Hamiltonian $(-P_\alpha)$ is given by (16) and it is a function of x, ϕ, P_x, P_ϕ and the intrinsic time parameter α . The canonical quantization of (16) yields

$$i\hbar \frac{\partial \psi}{\partial \alpha} = -\hat{P}_\alpha \psi = \frac{\hbar^2}{4M\beta x} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{n}{x} \frac{\partial \psi}{\partial x} \right) - \frac{\hbar^2}{2x} \frac{\partial^2 \psi}{\partial \phi^2} e^{-2\alpha} + V_e \psi, \quad (20)$$

where n is the operator ordering index and

$$V_e = V_e(x, \phi, \alpha) = \frac{M\beta}{x} (x^2 + 1)^2 + \frac{M}{x} (x^2 - 1) e^{2\alpha} + \frac{V(\phi)}{x} e^{4\alpha}. \quad (21)$$

Equation (20) is the correct Wheeler-deWitt equation corresponding to an action, containing curvature and curvature squared terms along with a minimally coupled scalar field. It looks similar to the Schrödinger equation, though it differs from that obtained by Pollock [14].

III. PROBABILITY AND CURRENT DENSITY

One of the most important feature observed in quantum mechanics is that, the state of a system is described by a wavevector belonging to an abstract Hilbert space and the norm of the wavevector must be positive definite or zero. This idea emerged from the interpretation of the probability density to describe a given state from the continuity equation which is obtained by using the Schrödinger equation. Probability interpretation follows from the simple mathematical appearance of the Schrödinger equation. No such interpretation of the probability density in general exists in quantum cosmology, when the action contains terms linear in the Ricci scalar coupled with some matter field. This is due to the fact that there is no time a priori in the Wheeler-deWitt equation in a gravitational theory described by the action (2) with $\beta = 0$. It is to be noted that the continuity equation along with the conventional notion of the probability density can only be introduced with the proper choice of the canonical variables in such a way that the Hamiltonian constraint is quadratic in the canonical momenta along with a term linear in momentum whose canonical coordinate acts as an intrinsic time variable. This type of canonical quantization is possible only when the action contains atleast a quadratic curvature term.

Equation (20) can be written as

$$i\hbar \frac{\partial \psi}{\partial \alpha} = \hat{H}_0 \psi, \quad (22)$$

where

$$\hat{H}_0 = \frac{\hbar^2}{4M\beta x} \left(\frac{\partial^2}{\partial x^2} + \frac{n}{x} \frac{\partial}{\partial x} \right) - \frac{\hbar^2}{2x} e^{-2\alpha} \frac{\partial^2}{\partial \phi^2} + \hat{V}_e(x, \phi, \alpha). \quad (23)$$

It is to be noted that \hat{H}_0 operator is hermitian and as a consequence we obtain the continuity equation, viz.

$$\frac{\partial \rho}{\partial \alpha} + \nabla \cdot \mathbf{J} = 0, \quad (24)$$

where ρ and \mathbf{J} are the probability and the current densities respectively for the choice of the operator ordering index $n = -1$. ρ and \mathbf{J} are given by $\rho = \psi^* \psi$ and $\mathbf{J} = (J_x, J_\phi, 0)$, where

$$J_x = \frac{i\hbar}{4M\beta x}(\psi^* \psi_x - \psi \psi_x^*), \quad (25)$$

$$J_\phi = -\frac{i\hbar e^{-2\alpha}}{2x}(\psi^* \psi_\phi - \psi \psi_\phi^*), \quad (26)$$

and

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial \phi}, 0 \right). \quad (27)$$

One can also find the continuity equation for other values of the operator ordering index but with respect to a new variable y which is functionally related to x only. It should be mentioned here that the continuity equation in quantum cosmology was presented by Sanyal and Modak [2] for the first time in a recent publication. In the present work we have obtained the continuity equation once again. Since the above probability and the current densities are dynamical quantities, therefore the wavefunction and its derivatives should remain finite at all epoch of the evolution of the Universe, if and only if there are no singularities in the domain of quantum cosmology.

Further in analogy with the quantum mechanics it is noted that the variable α in equation (20) can be identified as the time parameter in quantum cosmology. The variables $x(= \dot{\alpha})$ and ϕ act as spatial coordinate variables in the context of quantum cosmology. The coordinate variable $x(= \dot{\alpha})$, which is just the expansion parameter and the Hamiltonian operator \hat{H}_0 diverges at the bounce of the Universe, ie. at $\dot{\alpha} = 0$.

IV. EFFECTIVE POTENTIAL AND ITS EXTREMIZATION

The effective potential given by equation (21) is found to be assymmetric with respect to the expansion parameter (x) , ie. $V_e(-x) = -V_e(+x)$ and it is also unstable both at the short range and the long range values of x . The significance of the effective potential becomes clear when it is extremized, yielding interesting results. In the weak energy limit, ie. when the deBroglie wavelength is much larger in comparison with the average separation (ie. far below the Planck scale) the contribution of the kinetic energy is sufficiently small with respect to the potential energy in the Hamiltonian. At this regime the Hamiltonian is almost dominated by the potential energy. Now the extremum of the effective potential V_e given by equation (21), with respect to x yields, keeping ϕ and the time parameter α fixed,

$$\beta(3x^4 + 2x^2 - 1) + (x^2 + 1)e^{2\alpha} - \frac{e^{4\alpha}}{M}V(\phi) = 0. \quad (28)$$

If $V(\phi)$ has got no extremum then in the limit $\beta \rightarrow 0$, ie. in the absence of the curvature squared term, equation (28) takes the following form

$$x^2 + 1 = \dot{\alpha}^2 + 1 = \frac{e^{2\alpha}}{M}V(\phi), \quad (29)$$

which is exactly the classical constraint equation in the Einstein's gravity, along with a scalar field, apart from a kinetic energy part corresponding to the scalar field, that we have already assumed to be small. Further, for the action dominated by the curvature squared term, ie. at $\beta \rightarrow \infty$, we have

$$x^2 + 1 = 0, \quad (30)$$

or,

$$x^2 - \frac{1}{3} = 0. \quad (31)$$

These pair of equations have already been obtained in [2]. Here, we have just been able to recover them in the limit $\beta \rightarrow \infty$, ie. at the epoch of the evolution of the Universe dominated by the curvature squared term. These equations imply that the extremum of the effective potential V_e is at a coordinate position $x = \dot{\alpha}$, satisfying either equation (30), which is the vacuum Einstein's equation that admits Euclidean wormhole solution, or equation (31), which is the epoch at which the Universe admits a solution, given by $a = \frac{(t-t_0)}{\sqrt{3}}$, 'a' being the scale factor in proper time 't'. For this solution the horizon radius r_H is proportional to $\ln(t)$, and so as $t \rightarrow 0, r_H \rightarrow \infty$, and thus the horizon problem is solved. The effective potential V_e diverges in both the directions of $x \rightarrow 0$ and $x \rightarrow \infty$, even for $\beta \rightarrow \infty$. Further, it can be shown that $\frac{\partial^2 V_e}{\partial x^2} > 0$, for condition (31), ie. V_e has got a minimum. These altogether imply that the Universe, while sitting at the minimum of the effective potential, will expand at the rate a proportional to t , solving the horizon problem.

Now, had there been an extremum of the scalar field potential $V(\phi)$, then extremizing the effective potential V_e , given in equation (21), with respect to ϕ , keeping x, α fixed, we get

$$\frac{e^{4\alpha}}{Mx} \frac{dV}{d\phi} = 0. \quad (32)$$

The condition that a function (V_e) of two variables x , and ϕ (not considering the time parameter α) has got a minimum, is

$$\frac{\partial^2 V_e}{\partial x^2} \frac{\partial^2 V_e}{\partial \phi^2} - \frac{\partial^2 V_e}{\partial x \partial \phi} > 0. \quad (33)$$

The left hand side in the present context turns out to be

$$\frac{2e^{4\alpha}}{Mx_0^2} [e^{2\alpha} + 2\beta(3x_0^2 + 1)] \frac{d^2 V(\phi)}{d\phi^2} \Big|_{at\phi=\phi_0} \quad (34)$$

which is positive definite for $\frac{d^2 V(\phi)}{d\phi^2} \Big|_{at\phi=\phi_0} > 0$, ie. for a minimum of the scalar field potential. Therefore the effective potential has got an extremum located at the configuration space variables $\phi = \phi_0$ and $x = x_0$, determined by equation (28).

It can further be shown that $\frac{\partial^2 V_e}{\partial x^2} \Big|_{\phi, \alpha = \text{constant}} > 0$, which means that the effective potential is truly a minimum at the location of the configuration space variables $\phi = \phi_0, x = x_0$, determined by equation (28), where $V(\phi) = V(\phi_0)$. Further, if we choose the minimum of the scalar field potential $V(\phi_0) = 0$, then equation (28) reduces to either

$$x^2 + 1 = 0 \quad (35)$$

or,

$$\beta(3x^2 - 1) + e^{2\alpha} = 0. \quad (36)$$

these pair of equations again imply that the Universe is sitting at the minimum of the effective potential V_e , at the locations of the configuration space variables ϕ_0 and $x = x_0$, satisfying either equation (35), which is the vacuum Einstein's equation admitting Euclidean wormhole solution, or equation (36), that admits an oscillatory solution, in the form

$$a^2 = \beta \sin^2\left(\frac{t-t_0}{\sqrt{3}\beta}\right), \quad (37)$$

where, as already mentioned, 'a' is the scale factor in proper time 't'.

V. CONCLUDING REMARKS

In summary, we have observed that, different choice of the auxiliary variable, leads to different inequivalent quantum dynamics, keeping the classical field equations unchanged. Hence, we have suggested a principle of choosing the correct auxiliary variable and the true degree of freedom, which would lead to the correct and unique quantum description. For this we have modified Boulware et al's principle [1] by adding only one statement at the beginning that, before choosing such variables one has to remove all removable total derivative terms from the action. That this would lead to the correct quantum description, has been proved in a toy model, in a recent publication by

Sanyal and Modak [2]. In addition, it has been shown that the quantum dynamics obtained by this principle leads to certain desirable features viz. quantum mechanical probability and current density interpretation from the continuity equation of quantum cosmology and an effective potential whose extremum yields classical field equations. It seems that a quantum mechanical probability interpretation is possible, only if the Einstein-Hilbert action in the Robertson-Walker minisuperspace model is modified by atleast curvature squared term, which appears in the one loop correction of perturbative quantum field theory in curved space time, and is the most dominant term in the quantum domain. Thus we may conclude that, quantum mechanical probability interpretation of quantum cosmology is a generic feature of curvature squared gravity, at least in the Robertson-Walker minisuperspace model.

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